Lecture Note 02: *Static Optimization*

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1.1 Derivatives

Definition 1.1 Derivative

The **derivative** of the function f at x is defined as

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The **derivative of a function provides its slope at a point** x. It represents **the change in** f **when** x **increase by an infinitesimal amount**. We'll commonly use the notation $f'(x)$ to denote the derivative of $f(x)$ with respect to x. The derivative of a function evaluated at an specific value c is denoted by $f'(x)|_{x=c}$ or $\frac{df(x)}{dx}|_{x=c}$.

Using the previous definition we can *derive* some common rules that will make our life easier when dealing with functions that will appear in most economic problems. The next table summarize this common set of rules, you can see the Appendix A for proof of each rules (it is a good exercise to do this proofs yourself at least once to *warm up*).

Derivative rules

Name	**Function**	**Derivative**
Constant	a	0
Linear	x	1
Exponential	e^x	e^x
Logarithms	$\ln(x)$	$\frac{1}{x}$
Power Rule	x^n	nx^{n-1}
Multiplication by a constant	$af(x)$	$af'(x)$
Sum Rule	$f(x) + g(x)$	$f'(x) + g'(x)$
Product Rule	$f(x)g(x)$	$f'(x)g(x) + f(x)g'(x)$
Quotient Rule	$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$
Chain Rule	$f(g(x))$	$f'(g(x))g'(x)$

When the derivative of a function exists for all the domain in which the function is defined then we say that the function is **differentiable**.

1.2 Higher order derivatives

Definition 1.2 *nth* derivative

Let n be an integer. The **nth derivative** of a function $f(x)$ is obtained by sequentially taking the derivative of the original function n times.

In most economic problems we are going to be interested in the **first derivative** $f'(x)$ and the **second derivative** $f''(x)$ of a function. This is because most economic problems are about finding *extreme* values of a function. In the next section we are going to find rules that allow us to find and verify the extreme values of a function by computing the first and second derivative.

1.3 Extreme values

Let a and b be two constants. If $a < b$, the set of numbers between a and b that do not include a and b is an **open interval**. We denote open intervals with parentheses (a, b), another notation for the same interval is $a < x < b$. If the values a and b are included in the interval then we have a **closed interval**. This are denoted with brackets $[a, b]$ or $a \leq x \leq b$. Finally, a **half-open interval** do not include b and a **half-closed interval** exclude a. This can alternatively be written as $a \leq x < b$ and $a < x \leq b$ respectively. An **interior point** c is a point in an interval such that there are values greater or smaller than c that are also in the interval.

The **extreme value**, or extremum, of a function is the largest (maximum) or smallest (minimum) value of the function. The minimum/maximum values of a function defined in the entire domain of the function are called **global extreme values**. The minimum/maximum values of a function defined in a subset of the domain (neighborhood around a point) in which the function is defined are called **local extreme values**. Is common to refer to extreme values as critical or stationary points.

Theorem 1.1 Extreme value theorem for continuous function

If $f(x)$ is continuous in the closed interval $[a, b]$, then the **maximum** and **minimum** values of $f(x)$ exist.

The last theorem is also known as the **Max-min theorem** (not to be confused with the Min-max theorem in Linear Algebra).

1.4 First and second order conditions

Proposition 1.1 First Order Condition (for interior points)

The **first order condition** is a *necessary* condition for finding extreme values. It states that a point x^* in the closed interval $[a, b]$, such that $f'(x)|_{x^*} = 0$ is a local extreme value.

Proof. Assume a point x^* is an interior point such that $f(x)$ is maximized at x^*. By definition of maximum it has to be the case that $f(x^*) - f(x) \geq 0$ for $x \in [a, b]$. Then,

$$\begin{aligned}
f(x^*) - f(x) &\geq 0 \\
\frac{f(x^*) - f(x)}{x - x^*} &\geq 0 \\
\lim_{x \to x^{*+}} \frac{f(x^*) - f(x)}{x - x^*} &\geq 0
\end{aligned}$$

and,

$$\lim_{x \to x^{*-}} \frac{f(x^*) - f(x)}{x - x^*} \leq 0$$

Note that the sign changes because $x - x^* < 0$ when approaching from values smaller than x^*. Combining this two results,

$$\begin{aligned}
\lim_{x \to x^*} \frac{f(x^*) - f(x)}{x - x^*} &= 0 \\
f'(x)|_{x^*} &= 0
\end{aligned}$$

∎

You can easily verify that this is also true for a minimum.

Note that the last result was **only true for interior points**. This is so because it may be the case that the function $f(x)$ is maximized (minimized) at one of the endpoints of the interval $[a, b]$, and at those points the function is not differentiable (the derivative is not defined because the function is not defined for values outside of the interval and the limits from both sides cannot be computed). Assuming that the function is differentiable in the interval $[a, b]$, this implies that the only values that an extreme value can be found are:

- $f'(x)|_{x^*} = 0$ if x^* is an interior point.
- Endpoints of the interval $[a, b]$.

Keep in mind that, and this is very important, the first order condition is *just* a necessary condition for an extreme value and the only information it conveys is that "*if x^* is interior and* *the first derivative is equal to zero then its an extreme value*". Therefore, the first order condition cannot be used to tell: (1) if x^* is a global extreme value, and/or, (2) if x^* is a minimum or a maximum. The first order condition is usually abbreviated as **FOC**.

Proposition 1.2 Second Order Condition (for interior points)

The **second order condition** is a *sufficient* condition for finding extreme values. It states that a point x^* in the closed interval $[a, b]$, such that $f'(x)|_{x^*} = 0$ can be classified as a local minimum (maximum) in the given interval if:

- $f''(x)|_{x^*} > 0$ then x^* is a local minimum.
- $f''(x)|_{x^*} < 0$ then x^* is a local maximum.

Proof. Assume a point x^* is an interior point such that $f''(x)|_{x^*} > 0$. Then, it follows from the definition of derivative that:

$$\begin{aligned} f''(x)|_{x^*} &= \lim_{x \to x^*+} \frac{f'(x^*) - f'(x)}{x - x^*} > 0 \\ &= \lim_{x \to x^*+} \frac{-f'(x)}{x - x^*} > 0 \end{aligned}$$

and,

$$\begin{aligned} f''(x)|_{x^*} &= \lim_{x \to x^*-} \frac{f'(x^*) - f'(x)}{x - x^*} > 0 \\ &= \lim_{x \to x^*-} \frac{-f'(x)}{x - x^*} > 0 \end{aligned}$$

Recall that $f'(x)|_{x^*} = 0$. Also, note that in the first limit the denominator is positive because we are taking the limit *from the right* and $x > x^*$. This in turn imply that for the inequality to be satisfied $\lim_{x \to x^*+} f'(x) < 0$. Similarly, for the second case we have that $\lim_{x \to x^*-} f'(x) > 0$. That is, when approaching the extreme value from the right, the function is decreasing; this implies that the slope is positive to the right of the extreme value. Also, when approaching the extreme value from the left, the function is decreasing, this implies that the slope is negative to the left of the extreme value. Therefore, if the function is decreasing from the left and increasing from the right then at the extreme value the function is minimized. You can easily verify that for the case of $f''(x)|_{x^*} < 0$ the function is maximized. ■

Similar to the case of the FOC, the previous result **only applies for the case of interior**

points. It follows that if $[a, b]$ are interior points the first derivative is not defined and, therefore, the second derivative is also not defined. This implies that we have to be careful when using the second order condition, or **SOC**, for finding the minimum or maximum of a function. In particular we can summarize the rules for finding and classifying extreme values using the FOC and SOC tests as follows.

Classification of local extreme values

Let x^* be an interior point in the interval $[a, b]$, then:

Classification	Fist Derivative	Second Derivative
Local Minimum	$f'(x)\|_{x^*} = 0$	$f''(x)\|_{x^*} > 0$
Local Maximum	$f'(x)\|_{x^*} = 0$	$f''(x)\|_{x^*} < 0$
Not a local minimum/maximum	$f'(x)\|_{x^*} = 0$	$f''(x)\|_{x^*} = 0$

If $x^* = b$ is the upper bound of the interval, then

Classification	Limit
Local Minimum	$\lim_{x \to x^*-} f'(x) < 0$
Local Maximum	$\lim_{x \to x^*-} f'(x) > 0$

If $x^* = a$ is the lower bound of the interval, then

Classification	Limit
Local Minimum	$\lim_{x \to x^*+} f'(x) < 0$
Local Maximum	$\lim_{x \to x^*+} f'(x) > 0$

1.5 Examples of finding and classifying local extreme values

Example 1. *Find the extreme value of the following functions and classify them as local maximum or local minimum.*

1. $f(x) = x^2$
2. $f(x) = x^3 + -12x^2 + 36x + 8$
3. $f(q) = q^2 + -5q + 8$
4. $\pi(q) = q^{0.5} - 2q$
5. $f(x) = a$

First we need to find the extreme values using the the first derivative rule.

1. For $f(x) = x^2$

$$\begin{aligned} f'(x) &= 0 \\ 2x &= 0 \\ x^* &= 0 \end{aligned}$$

2. For $f(x) = x^3 + -12x^2 + 36x + 8$

$$\begin{aligned} f'(x) &= 0 \\ 3x^2 - 24x + 36 &= 0 \\ 3(x-6)(x-2) &= 0 \end{aligned}$$

$$x_1^* = 2 \qquad \textit{and} \qquad x_2^* = 6$$

3. For $f(q) = q^2 + -5q + 8$

$$\begin{aligned} f'(q) &= 0 \\ 2q - 5 &= 0 \end{aligned}$$

$$q^* = \frac{5}{2}$$

4. For $\pi(q) = q^{0.5} + -2q$

$$\begin{aligned} f'(q) &= 0 \\ 0.5q - 2 &= 0 \end{aligned}$$

$$q^* = 4$$

5. For $f(x) = a$

$$f'(x) = 0$$

Because $f'(x) = 0 \ \forall x$ then all x are local extreme values.

To classify the local extreme values we need to find the value of the second derivative at the extreme values.

1. *For* $f(x) = x^2$

$$\begin{aligned} f'(x) &= 2x \\ f''(x) &= 2 \\ f''(0) &= 2 \end{aligned}$$

Because $f''(2) > 0$ *we know that* $x^* = 2$ *is a local minimum.*

2. *For* $f(x) = x^3 + -12x^2 + 36x + 8$

$$\begin{aligned} f'(x) &= 3x^2 - 24x + 36 \\ f''(x) &= 6x - 24 \\ f''(2) &= -12 \\ f''(6) &= 12 \end{aligned}$$

Because $f''(2) < 0$ *we know that* $x^* = 2$ *is a local maximum and because* $f''(6) > 0$ *we know that* $x^* = 6$ *is a local minimum.*

3. *For* $f(q) = q^2 + -5q + 8$

$$\begin{aligned} f'(q) &= 2q - 5 \\ f''(q) &= 2 \\ f''(\frac{5}{2}) &= 2 \end{aligned}$$

Because $f''(\frac{5}{2}) > 0$ *we know that* $x^* = \frac{5}{2}$ *is a local minimum.*

4. *For* $\pi(q) = q^{0.5} + -2q$

$$\begin{aligned} f'(q) &= 0.5q - 5 \\ f''(q) &= 0.5 \\ f''(4) &= 0.5 \end{aligned}$$

Because $f''(4) > 0$ *we know that* $x^* = 4$ *is a local minimum.*

5. *For $f(x) = a$*

$$\begin{aligned} f'(x) &= 0 \\ f''(x) &= 0 \end{aligned}$$

Every x satisfy the first order condition, yet for there is no x such that $f''(x) > 0$ nor $f''(x) < 0$; therefore, there are no local maximum nor minimum.

1.6 Conditions for global extreme values

Until now we have characterized the necessary and sufficient conditions to classify local extreme values. In order to guarantee that the extreme values are in fact global extreme values we need to impose some extra assumptions to the underlying function besides being continuous and twice differentiable for all the points in the domain of the function.

Definition 1.3 Concave function

A function $f(x)$ is **concave** if for the closed interval $[a, b]$:

$$f(\alpha x + (1-\alpha)x') \geq \alpha f(x) + (1-\alpha) f(x')$$

for all x and x' in the interval and all $\alpha \in [0, 1]$. If the inequality is strict then we say that the function is **strictly concave**.

Keep in mind that the previous condition has to be satisfied *for all* values x and x' in the interval, i.e. for *every* pair of points in the domain of the function, the value of the function evaluated in the linear combination of the two points is greater than the linear combination of the function at the two original points. A **convex** function can be defined similarly changing the direction of the inequality.

There is a link between the definition of concavity (convexity) and the first derivative of a function.

Proposition 1.3 Concavity and first derivative

$f(x)$ is a concave function in the closed interval $[a, b]$, if and only if

$$f(x) \leq f(x') + f'(x')(x - x')$$

for all values x and x' in the interval $[a, b]$.

Proof. Start with the defintion of concavity,

$$\begin{aligned}
f(\alpha x + (1-\alpha)x') &\geq \alpha f(x) + (1-\alpha) f(x') \\
f(x' + \alpha(x - x')) &\geq f(x') + \alpha[f(x) - f(x')] \\
\frac{f(x' + \alpha(x - x')) - f(x')}{\alpha} &\geq f(x) - f(x') \\
\lim_{a \to 0} \frac{f(x' + \alpha(x - x')) - f(x')}{\alpha} &\geq \lim_{a \to 0} [f(x) - f(x')] \\
f'(x')(x - x') + f(x') &\geq f(x)
\end{aligned}$$

■

In the last step we use a Taylor series to compute limit in the left hand side. There is an analogous condition for convex function; again, to find the link between convex functions and the first derivative you only need to change the sign of the inequality and follow the same steps as the previous proof.

Theorem 1.2 Global maximum

If $f(x)$ is concave then a extreme value x^* in the domain of the function is a **global maximum**.

Proof. Using the condition for concavity evaluated at x^*,

$$f(x) \leq f(x^*) + f'(x^*)(x - x^*)$$

Because x^* is an extreme value we know that $f'(x^*) = 0$. Then the previous condition becomes,

$$f(x) \leq f(x^*)$$

Because the definition of concavity is for *every* pair of points in the domain; then it has to be the case that the value of the function at x^* is greater or equal to the value of the function at any other point in the domain; that is, x^* is a global maximum. ■

Similarly we can prove that assuming that function is convex guarantees that an extreme value is a global minimum. Note that previous theorem does not guarantee that the global maximum x^* is unique. To guarantee **uniqueness** we need to assume that the function is *strictly* concave.

Theorem 1.3 Strict concavity and uniqueness

Let x^* be the global maximum of an strictly concave function then x^* is the unique global maximum.

Proof. We are going to proof this by contradiction. Assume that simultaneously x and x' are global maximizers in the interval $[a, b]$. Now, define $x_\alpha = \alpha x + (1 - \alpha)x'$ for $\alpha \in [0, 1]$. By strict concavity

$$f(x_\alpha) > \alpha f(x) + (1 - \alpha)f(x') = f(x) = f(x')$$

This contradicts the assumption that x and x' are both global maximizers. ■

Not all optimization problems are about concave functions. In several cases you will find less restrictive assumptions about the behavior of the function. Often you will come across with **quasiconcave** and **quasiconvex** functions.

Definition 1.4 Quasiconcave function

A function $f(x)$ is **quasiconcave** if for all x and x' in $[a, b]$,

$$f(\alpha x + (1 - \alpha)x') \geq \min\{f(x), f(x')\}$$

where $\alpha \in [0, 1]$. A function is **strictly quasiconcave** if the inequality is strict.

Similarly, changing the sign of the inequality we can define **quasiconvex** and **strictly quasiconvex** functions. There is also a relation among quasiconcavity and the first derivative.

Proposition 1.4 Quasiconcavity and first derivative

$f(x)$ is a quasiconcave function in the closed interval $[a, b]$, if and only if

$$f'(x)(x' - x) \geq 0 \quad \text{whenever} \quad f(x') \geq f(x)$$

for all values x and x' in the interval $[a, b]$.

Proof. Start with the defintion of quasiconcavity and $f(x') \geq f(x)$,

$$\begin{aligned}
f(\alpha x + (1-\alpha)x') &\geq \min\{f(x), f(x')\} \\
f(\alpha x + (1-\alpha)x') &\geq f(x) \\
\frac{f(\alpha x + (1-\alpha)x') - f(x)}{1-\alpha} &\geq 0 \\
\lim_{\alpha \to 1} \frac{f(\alpha x + (1-\alpha)x') - f(x)}{1-\alpha} &\geq 0 \\
f'(x)(x' - x) &\geq 0
\end{aligned}$$

∎

We can prove that changing the sign this result will hold for quasiconvex functions. Quasiconcave functions do *not* guarantee that a an extreme value is a global critical value. On the other hand, strict quasiconcavity does.

Theorem 1.4 Strict quasiconcavity and global maximum

In the closed interval $[a, b]$. If $f(x)$ is strictly quasiconcave and x^* is an extreme value in the domain of the function then x^* is a global maximum.

Proof. We are going to proof this by contradiction. That is, we are going to show that if $f(x)$ is strictly quasiconcave and x^* is an extreme value for all x in the interval $[a, b]$, $f(x) > f(x^*)$ is a contradiction; implying that $f(x^*) \geq f(x)$ for all x - which is the definition of a global maximum -.

Assume that $f(x)$ is strictly quasiconcave and there is an x such that $f(x) > f(x^*)$. By strictly quasiconcavity it has to be true that:

$$\begin{aligned}
f(\alpha x + (1-\alpha)x^*) &> f(x^*) \\
\frac{f(\alpha x + (1-\alpha)x^*) - f(x^*)}{\alpha} &> 0 \\
\lim_{\alpha \to 0} \frac{f(\alpha x + (1-\alpha)x^*) - f(x^*)}{\alpha} &> 0 \\
f'(x^*)(x - x^*) &> 0
\end{aligned}$$

If x^* is an extreme value then $f'(x^*) = 0$ and the last line will never be satisfied. Therefore, there are no $x \in [a, b]$ such that $f(x) > f'(x^*)$. This implies that x^* is a global maximum. ■

Analogous to this we can prove that a local minimum is the global minimum of a strictly quasiconvex function. Moreover, if x^* is the global maximum of an strictly quasiconcave function we know that x^* is the **unique** maximum.

Theorem 1.5 Uniqueness and strict quasiconcavity

Let x^* be the global maximum of an strictly quasiconcave function then x^* is the unique global maximum.

Proof. We are going to proof this by contradiction. Assume that simultaneously x and x' are global maximizers in the interval $[a, b]$. Now, define $x_\alpha = \alpha x + (1-\alpha)x'$ for $\alpha \in [0, 1]$. By strict quasiconcavity

$$f(x_\alpha) > \min\{f(x), f(x')\} = f(x) = f(x')$$

This contradicts the assumption that x and x' are both global maximizers. ■

As usual, uniqueness of global minimum is guarantee for strictly quasiconvexity; to prove it you just need to change the inequality sign. The next table summarize the main results on global extreme values.

Important results on global extreme values

For maximum extreme value:

Assumption on $f(x)$	**Global maximum**	**Uniqueness**
Concave	Guaranteed	Not Guaranteed
Strictly Concave	Guaranteed	Guaranteed
Quasioncave	Not Guaranteed	Not Guaranteed
Strictly Quasiconcave	Guaranteed	Guaranteed

For minimum extreme value:

Assumption on $f(x)$	**Global minimum**	**Uniqueness**
Convex	Guaranteed	Not Guaranteed
Strictly Convex	Guaranteed	Guaranteed
Quasionvex	Not Guaranteed	Not Guaranteed
Strictly Quasiconvex	Guaranteed	Guaranteed

Example 2. *Determine if the following functions are concave, convex or not.*

1. $f(x) = \sqrt{x}$

2. $f(x) = a + bx$

We need to verify that $f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha)f(y)$ *for every pair* (x, y).

1. *For* $f(x) = \sqrt{x}$

$$
\begin{aligned}
f(\alpha x + (1-\alpha)y) &\geq \alpha f(x) + (1-\alpha)f(y) \\
\sqrt{\alpha x + (1-\alpha)y} &\geq \alpha\sqrt{x} + (1-\alpha)\sqrt{y} \\
\alpha x + (1-\alpha)y &\geq (\alpha\sqrt{x} + (1-\alpha)\sqrt{y})^2 \\
\alpha x + (1-\alpha)y &\geq \alpha^2 x + 2\alpha(1-\alpha)\sqrt{xy} + (1-\alpha)^2\sqrt{y} \\
\alpha(1-\alpha)(x+y) &\geq 2\alpha(1-\alpha)\sqrt{xy} \\
x + y &\geq 2\sqrt{xy} \\
(x+y)^2 &\geq 4xy \\
x^2 + 2xy + y^2 &\geq 4xy \\
x^2 + y^2 &\geq 2xy \\
(x-y)^2 &\geq 0
\end{aligned}
$$

Because for all x *and* y, $(x-y)^2 \geq 0$ *is true, then* $f(x)$ *is a concave function.*

2. *For* $f(x) = a + bx$

$$
\begin{aligned}
f(\alpha x + (1-\alpha)y) &= a + b(\alpha x + (1-\alpha)y) \\
&= \alpha a + (1-\alpha)a + b\alpha x + b(1-\alpha)y \\
&= \alpha(a+bx)(1-\alpha)(a+by) \\
&= \alpha f(x)(1-\alpha)f(y)
\end{aligned}
$$

Therefore, $f(x) = a + bx$ *is both concave and convex, it is also strictly concave (convex).*

2 Unconstrained optimization of functions with several variables

2.1 Gradients and Hessians

In the subsequent part of the lecture note x will denote a **vector** instead of a single variable. An element of x is going to be denoted by the subscript i. Therefore, $x_1 \in x$ is the first element of the vector x. Similarly, $f(x)$ now is going to represent a function that takes the values of several variables and return a real value $\mathbb{R}^n \to \mathbb{R}$. A **partial derivative** is the derivative of $f(x)$ with respect to x_i. A partial derivative is denoted by $\dfrac{\partial f(x)}{\partial x_i}$, the notation $f_i'(x)$ is also common.

Definition 2.1 Gradient

A **gradient** $\nabla f(x)$ is a vector-valued function $\mathbb{R}^n \to \mathbb{R}^n$ such that the element i of $\nabla f(x)$ is the first derivative of $f(x)$ with respect to x_i. Therefore,

$$\nabla f(x) = (\partial f(x)/\partial x_1, \partial f(x)/\partial x_2, ..., \partial f(x)/\partial x_n)$$

A gradient is just a list of the n partial derivatives of the function with respect to each variable in the function. The gradient is the extension of the concept of derivative to functions with more than one variable. In the case of $n = 1$ the gradient represents the slope of the function, if $n > 1$ a gradient *points* in the direction of the greatest increase of the function and its magnitude is the slope of the graph in that direction. Intuitively we can think that like in the case for $n = 1$ when the slope is zero we have an extreme value, then for $n > 1$ if the magnitude of the gradient is zero then the we can also find an extreme value for the function.

Definition 2.2 Hessian

The **hessian** $H(x)$ of a function $f(x)$ is an square matrix such that the element i, j is $H_{i,j} = \dfrac{\partial^2 f(x)}{\partial x_i \partial x_j}$, therefore

$$H = \begin{bmatrix} c_{11} \end{bmatrix}$$

The concept of Hessian is the extension of the idea of second derivative for $n > 1$. We are going to be using the Hessian in the same fashion that we use the second derivative to find sufficient conditions to classify extreme values as local minimum or local maximum.

2.2 Finding and classifying extreme values of functions with n variables

Proposition 2.1 First Order Condition for n variables

The **first order condition** is a necessary condition for finding extreme values. A vector x^* in the closed set A, such that $\nabla f(x) = 0$ is a local extreme value.

Proof. Assume that x^* is a local maximum. Consider the vector a such that all elements of a are equal to zero except for $a_i > 0$.

Consider $f(x^*) \geq f(x^* + \alpha a)$ for $\alpha \in [0, 1]$. Then,

$$\begin{aligned}
f(x^*) &\geq f(x^* + \alpha a) \\
0 &\geq f(x^* + \alpha a) - f(x^*) \\
0 &\geq \frac{f(x^* + \alpha a) - f(x^*)}{\alpha} \\
0 &\geq \lim_{\alpha \to 0} \frac{f(x^* + \alpha a) - f(x^*)}{\alpha} \\
0 &\geq a f_i'(x^*)
\end{aligned}$$

Because $a_i > 0$, we know that the last line implies $f_i'(x^*) \leq 0$.

Consider $f(x^*) \geq f(x^* - \alpha a)$ for $\alpha \in [0, 1]$. Then,

$$\begin{aligned}
f(x^*) &\geq f(x^* - \alpha a) \\
0 &\geq f(x^* - \alpha a) - f(x^*) \\
0 &\geq \frac{f(x^* - \alpha a) - f(x^*)}{\alpha} \\
0 &\geq \lim_{\alpha \to 0} \frac{f(x^* + \alpha a) - f(x^*)}{\alpha} \\
0 &\geq -a f_i'(x^*)
\end{aligned}$$

Because $a_i > 0$, we know that the last line implies $f_i'(x^*) \geq 0$. Combining this two results we get that $f_i'(x^*) = 0$. And because this is true for all $i \in N$. This implies that $\nabla f(x^*) = 0$. ■

The last proof can also be used to show that in a local minimum $\nabla f(x) = 0$, just need to change the inequality sign. This imply that we now have a rule for finding extreme values, yet we still need a rule to classify them into local maximum or local minimum.

Proposition 2.2 Second Order Condition for n variables

The **first order condition** is a sufficient condition for finding and classifying extreme values. A vector x^* in the closed set A, such that $H(x^*)$ is negative semidefinite is a local maximum.

Proof. Assume that x^* is a local maximum. Consider the vector a such that all elements of a are equal to zero except for $a_i > 0$.

Consider $f(x^*) \geq f(x^* + \alpha a)$ for $\alpha \in [0, 1]$. Then,

$$\begin{aligned} f(x^*) &\geq f(x^* + \alpha a) \\ 0 &\geq f(x^* + \alpha a) - f(x^*) \\ 0 &\geq \nabla f(x^* + \alpha a) - \nabla f(x^*) \\ 0 &\geq \lim_{\alpha \to 0} \frac{\nabla f(x^* + \alpha a) - \nabla f(x^*)}{\alpha} \\ 0 &\geq a' H(x^*) a \end{aligned}$$

The last line is the definition of a $H(x)$ being negative semidefinite. The substitution of the last line is obtained using a Taylor Series expansion, note that using the Taylor Series $\nabla f(x^* + \alpha a) = \nabla f(x^*) + \alpha a' H(x^*) a +$ higher order terms. Because the higher order terms end up being multiplied by α when taking the limit $\alpha \to 0$ they become zero.

■

For local minimum we just need to change the sign, therefore: for a local minimum the second order condition is that the hessian is positive semidefinite.

With the last two results we can write out rules on the necessary and sufficient conditions when finding local extreme values.

Classification of local extreme values for functions with n variables

Let x^* be an interior point in the closed set A, then:

Classification	Gradient	Hessian
Local Minimum	$\nabla f(x) = 0$	Positive semidefinite
Local Maximum	$\nabla f(x) = 0$	Negative semidefinite
Not a local minimum/maximum	$\nabla f(x) = 0$	Not positive/negative semidefinite

If $x^* = b$ is the upper bound of the closed set, then

Classification	Limit of the gradient
Local Minimum	$\lim_{x \to x^{*-}} \nabla f(x) < 0$
Local Maximum	$\lim_{x \to x^{*-}} \nabla f(x) > 0$

If $x^* = a$ is the lower bound of the closed set, then

Classification	Limit of the gradient
Local Minimum	$\lim_{x \to x^{*+}} \nabla f(x) < 0$
Local Maximum	$\lim_{x \to x^{*+}} \nabla f(x) > 0$

2.3 Conditions for global extreme values of functions with n variables

We are going to be use the definitions of concavity (convexity) and quasiconcavity (quasiconvexity) from the previous chapter to find when this assumptions can guarantee the existence and/or uniqueness of global extreme values.

Theorem 2.1 Global maximum for functions with n variables

If $f(x)$ is concave then a extreme value x^* in the domain of the function is a **global maximum**.

Proof. Using the condition for concavity evaluated at x^*,

$$f(x) \leq f(x^*) + \nabla f(x^*)(x - x^*)$$

Because x^* is an extreme value we know that $\nabla f(x^*) = 0$. Then the previous condition becomes,

$$f(x) \leq f(x^*)$$

Because the definition of concavity is for *every* pair of points in the domain; then it has to be the case that the value of the function at x^* is greater or equal to the value of the function at any other point in the domain; that is, x^* is a global maximum. ■

Similarly we can prove that assuming that function is convex guarantees that an extreme value is a global minimum. Note that previous theorem does not guarantee that the global maximum x^* is unique. Recall from Theorem 1.3 that in order to guarantee uniqueness in a concave function we need to assume that the function is *strictly* concave. In fact, we can apply the same theorem for functions with n variables without making any change to (1.3). Conceptually x will be a vector instead of one single variable, but everything else is the same. Now, let's take a look at strictly quasiconcave functions.

Theorem 2.2 Strict quasiconcavity and global maximum of functions with n variables

In the closed set A. If $f(x)$ is strictly quasiconcave and x^* is an extreme value in the domain of the function then x^* is a global maximum.

Proof. We are going to proof this by contradiction. That is, we are going to show that if $f(x)$ is strictly quasiconcave and x^* is an extreme value for all x in A, $f(x) > f(x^*)$ is a contradiction; implying that $f(x^*) \geq f(x)$ for all x - which is the definition of a global maximum -.

Assume that $f(x)$ is strictly quasiconcave and there is an x such that $f(x) > f(x^*)$. By strictly quasiconcavity it has to be true that:

$$\begin{aligned}
f(\alpha x + (1-\alpha)x^*) &> f(x^*) \\
\frac{f(\alpha x + (1-\alpha)x^*) - f(x^*)}{\alpha} &> 0 \\
\lim_{\alpha \to 0} \frac{f(\alpha x + (1-\alpha)x^*) - f(x^*)}{\alpha} &> 0 \\
\nabla f(x^*)(x - x^*) &> 0
\end{aligned}$$

If x^* is an extreme value then $\nabla f(x^*) = 0$ and the last line will never be satisfied. Therefore, there are no $x \in A$ such that $f(x) > \nabla f(x^*)$. This implies that x^* is a global maximum. ■

Analogous to this we can prove that a local minimum is the global minimum of a strictly quasiconvex function. Moreover, if x^* is the global maximum of an strictly quasiconcave function we know that x^* is the unique maximum (See Theorem 1.5). Uniqueness is also guaranteed for extreme values of strictly quasiconvex functions.

3 Constrained optimization with equality constraints

Most optimization problems in economic are constrained optimization problems. In this problems we are going to have an agent that choose a variable (vector) x to maximize (minimize) an **objective function** such that a given set of constraints are satisfied. Optimization problems can be categorized into two groups depending on the type of constraint: (1) problems with equality constraints, and (2) problems with inequality constraints.

3.1 Equality constraint and setup

Definition 3.1 Equality Constraint

An **equality constraint** is a condition that most be satisfied in a constrained optimization problem such that:

$$g(x) = c$$

where c is a constant.

Definition 3.2 Maximization with equality constraints

Given a set of M equality constraints, an objective function $f(x)$ with domain $\mathbb{R}^N$. The problem $P(x)$ of **maximization with equality constraints** is defined as follows,

$$\begin{aligned} \underset{x \in \mathbb{R}^N}{\text{Max}} \quad & f(x) \\ \text{subject to,} \quad & g_1(x) = c_1 \\ & \vdots \\ & g_M(x) = c_M \end{aligned}$$

3.2 Lagrange's Method

Note that $N \geq M$ is a necessary condition to solving this problem. In order to find a solution to this type of problems we are going to use the **Lagrangian** method. But first we need to understand the theorem in which the method is based to know when it make sense to use it. First, let's define a Lagrangian.

Definition 3.3 Lagrangian

Given the maximization problem with equality constraints $P(x)$ the corresponding **Lagrangian** $\mathcal{L}(x, \lambda)$ is defined as:

$$\mathcal{L}(x, \lambda) = f(x) + \sum_{i=1}^{M} \lambda_i (c_i - g_i(x))$$

where λ is a set of M variables called **Lagrange multipliers**.

Theorem 3.1 Lagrangian Sufficiency Theorem

If x^* and λ^* are local maximizers of the Lagrangian $\mathcal{L}(x, \lambda)$, then x^* satisfied the constraints of $P(x)$ and locally maximized $f(x)$.

Proof. We need to prove that a local maximum of $\mathcal{L}(x, \lambda)$ implies that $f(x)$ is maximized given the equality constraints of $P(x)$. Assume that x^* and λ^* are local maximizers of the Lagrangian $\mathcal{L}(x, \lambda)$. Then by the FOC of unconstrained maximization we know that,

$$\begin{aligned} \nabla_x f(x^*) &= \sum_{i=1}^{M} \lambda_i \nabla_x g_i(x^*) \\ g_i(x^*) &= c_i \quad \forall i \in M \end{aligned}$$

The last set of FOC guarantees that the equality constraints in $P(x)$ are satisfied. And, because $\mathcal{L}(x^*, \lambda^*) = f(x^*)$ we know that x^* is a local maximum of $f(x)$. ■

Two things to note about Theorem 3.1. (1) It does not guarantee that x^* is a global maximum, and, (2) the theorem starts by assuming that x^* is a local maximum and just looking at the FOC does not imply that x^* is a local maximum. Let's deal with (2) first and then see what can we do about (1).

To be sure that x^* is a local maximum we need to use the SOC for unconstrained optimization on the Lagrangian. Let $H(x, \lambda)$ be the hessian of the Lagrangian, if $H(x, \lambda)$ is negative semidefinite then we know that x^* is a local maximum of $\mathcal{L}(x, \lambda)$, by Theorem 3.1, this implies that x^* is also a local maximum of $f(x)$.

From a geometric point of view note that the first set of FOC in the Lagrangian optimization problem implies that at x^* the linear combination of the gradients of the set of constraints is equal to the gradient of the objective function. The two gradients are going to be pointing in the same direction, but the values of λ has to be chosen in a way that the magnitude of $\nabla_x f(x^*)$ is equal to $\sum_{i=1}^{M} \lambda_i \nabla_x g_i(x^*)$. This will have an important economic interpretation that we'll see when doing economic applications.

3.3 Global maximum and uniqueness in maximization with equality constraints

Let's talk about global extreme values. We know from unconstrained maximization the conditions that need to be imposed on $f(x)$ guarantee that the extreme values are in fact global extreme values (concavity (strict or not necessary strict) and/or strict quasiconcavity). In the case of constrained optimization with equality constraints, because we can represent the optimization problem as an unconstrained optimization problem using the Lagrangian the results are going to be very similar, we only need to add some assumptions on the set of $g(x)$. We know that $\mathcal{L}(x, \lambda)$ is the combination of $f(x)$ and the set of $g(x)$.

You can verify that all concave functions are quasiconcave functions (same for strict). But not the other way around, not all quasiconcave are concave. Also, the nonnegative combination of concave (quasiconcave) functions is a concave (quasiconcave) function. Finally, the nonnegative combination of concave and quasiconcave functions is quasiconcave (this follows from all concave being quasiconcave).

Then, because the Lagrangian is just a combination of the objective function and the set of constraints we just need to apply the same rules that we applied to unconstrained optimization. Now, most of the applications of economic theory do not impose concavity (quasiconcavity) on $g(x)$ directly (yet most of the times $g(x)$ is a linear function and therefore strictly quasiconcave). Usually economist will say that the **solution set** should be a compact and convex set. That plus some assumption on the objective function will be sufficient for us to discuss the existence and uniqueness of global extreme values of the problem. But first, some important definitions.

Definition 3.4 Compact sets

A set A is **compact** if A is closed (it contains all its limit points) and bounded (the distance between every two points in the set is fixed).

Definition 3.5 Convex sets

A **convex set** A is a set such that,

$$x, x' \in A \text{ and } \alpha \in (0,1) \rightarrow \alpha x + (1-\alpha)x' = x_\alpha \in A$$

Is easy to verify that for two convex sets A, B all the set of elements that simultaneously $x \in A$ and $x \in B$ are a convex set.

Theorem 3.2 The intersection of convex sets is convex

If A and B are convex set then the combination C of A and B is convex.

Proof. Assume A and B to be convex sets. Let $x, x' \in A$ and $x, x' \in B$. Because A and B are convex sets then for $\lambda \in (0,1)$ it has to be the case that

$$\begin{aligned} \lambda x + (1-\lambda)x' &= x_\lambda \in A \\ \lambda x + (1-\lambda)x' &= x_\lambda \in B \end{aligned}$$

Let C be the set created from the elements that are in both A and B. Therefore, because every pair of points that is simultaneously in A and B (and in C) its linear combination x_λ is also in A and B (and in C). This imply that the linear combination of every pair of points in C is in C, therefore C is a convex set. ■

Definition 3.6 Upper (lower) contour set of a function

Given the relation $\geq$, a function $g : X \rightarrow \mathbb{R}^N$, the upper contour set of $g(x)$ are all the points $x \in R$ such that $g(x) \geq c$. Where c is a constant.

The upper contour set of a quasiconcave (strictly quasiconcave) function is convex (strictly quasiconvex). Therefore, if the upper contour set of $g(x)$ is convex then $g(x)$ is quasiconcave. Assuming that the upper contour set of the intersection of $g(x)$ is a convex (strict) set we know that the combination of $g(x)$ is a quasiconcave (strict) function.

Summary of conditions for global maximum and uniqueness in maximization problems with equality constraints

1. **Global Maximum**: An extreme value x^* is a global maximum of $\mathcal{L}(x, \lambda)$ if $f(x)$ is at least concave or strictly quasiconcave and the upper contour set of $g(x)$ is compact and convex.

2. **Uniqueness**: An extreme value x^* is the unique global maximum of $P(x)$ if $f(x)$ is at least strictly concave or strictly quasiconcave and the upper contour set of $g(x)$ is compact and convex.

4 Constrained optimization with inequality constraints

In most economic applications we won't have the luxury of assume that the constraints are satisfied with equality. Note that this was an essential part of proving that the FOC of the Lagrangian satisfied the constraints in the original maximization problem. In what follows I'll describe the method that we use to deal with maximization problems with inequality constraints.

4.1 Inequality constraints and setup

Definition 4.1 Inequality constraint

An **inequality constraint** is a condition that most be satisfied in a constrained optimization problem such that:

$$g(x) \leq c$$

where c is a constant.

Definition 4.2 Maximization with inequality constraints

Given a set of M equality constraints, an objective function $f(x)$ with domain $\mathbb{R}^N$. The problem $Q(x)$ of **maximization with inequality constraints** is defined as follows,

$$\begin{aligned} \underset{x\in\mathbb{R}^N}{\text{Max}} \quad & f(x) \\ \text{subject to,} \quad & g_1(x) \leq c_1 \\ & \vdots \\ & g_M(x) \leq c_M \end{aligned}$$

4.2 Karush-Kuhn-Tucker method

The **Karush-Kuhn-Tucker (KKT) method**, also know as the Kuhn-Tucker is a generalization of the Lagrange method for solving optimization problems with inequality. Let's discuss how the KKT method

Proposition 4.1 KKT method

If x^* and λ^* are local maximizers of the Lagrangian $\mathcal{L}(x, \lambda)$ and satisfy the conditions,

$$\lambda^*(c - g(x^*)) \geq 0$$
$$\lambda^* \geq 0$$

Then x^* is a local maximizer of the $Q(x)$ problem.

Proof. We need to proof that a local maximum of $\mathcal{L}(x, \lambda)$ that satisfy $\lambda^*(c - g(x^*)) \geq 0$ and $\lambda^* \geq 0$ implies that $f(x)$ is maximized given the inequality constraints of $Q(x)$. We are going to call this conditions the KT conditions. Assume x^* and λ^* are local maximizers of $\mathcal{L}(x, \lambda)$ and $g(x) = c$ for all the inequalities. Then this exactly the same problem as the $P(x)$ problem and we know that the conditions for a local maximizers using the Lagrange method satisfy $\lambda^*(c - g(x^*)) \geq 0$ and $\lambda^* \geq 0$. Now, if for some inequality $g(x) < c$ then with the Lagrange's method conditions we cannot guarantee that the pair (x^*, λ^*) is a local maximum. To verify this note that if (x^*, λ^*) is a local maximum it must be the case that $\mathcal{L}(x^*, \lambda^*) \geq \mathcal{L}(x, \lambda)$ for values sufficiently close to (x^*, λ^*), therefore:

$$\mathcal{L}(x, \lambda) \leq \mathcal{L}(x^*, \lambda^*) + \nabla_x \mathcal{L}(x^*, \lambda^*)(x^* - x) + \nabla_\lambda \mathcal{L}(x^*, \lambda^*)(\lambda^* - \lambda)$$

Because $\nabla_\lambda \mathcal{L}(x^*, \lambda^*) = c - g(x)$, we know that if we have a local maximum and $c - g(x^*) \geq 0$, then Lagrange method condition cannot guarantee that (x^*, λ^*) is an extreme value (because it does not imply that $\mathcal{L}(x, \lambda) \leq \mathcal{L}(x^*, \lambda^*)$). If $\lambda^* = 0$ in the case of $c - g(x^*) \geq 0$ then it is easy to verify that $\mathcal{L}(x, \lambda) \leq \mathcal{L}(x^*, \lambda^*)$ is satisfied. Therefore, if $\lambda^* = 0$ then $c - g(x) < 0$ and, if $\lambda^* \geq 0$ then $c - g(x) = 0$. Therefore, if (x^*, λ^*) satisfy the KT condition then they also solve the $Q(x)$ problem. ∎

Appendix A Derivatives rules

1 Derivative of a constant function

Let a be a constant. The derivative of the function $f(x) = a$ is $f'(x) = 0$

Proof.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{a-a}{h} \\ &= 0 \end{aligned}$$

■

2 Power rule

Let n be an integer. The derivative of the function $f(x) = x^n$ is $f'(x) = nx^{n-1}$

Proof.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h\to 0} \frac{(x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + ... + nxh^{n-1} + h^n) - x^n}{h} \\ &= \lim_{h\to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + ... + nxh^{n-1} + h^n}{h} \\ &= \lim_{h\to 0} nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + ... + nxh^{n-2} + h^{n-1} \\ &= nx^{n-1} \end{aligned}$$

■

Note that in the third line we use the **Binomial Theorem** to make the following substitution.

$$(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + ... + nxh^{n-1} + h^n$$

Also, the previous definition implies that the derivative of $f(x) = x$ is $f'(x) = 1$.

3 Derivative of the exponential function

The derivative of the function $f(x) = e^x$ is $f'(x) = e^x$

Proof.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} \\
&= \lim_{h\to 0} \frac{e^x(e^h - 1)}{h} \\
&= e^x \lim_{h\to 0} \frac{e^h - 1}{h} \\
&= e^x
\end{aligned}$$

∎

The last part is true because $\lim_{h\to 0} \dfrac{e^h - 1}{h} = 1$.

4 Derivative of the natural logarithm

The derivative of the function $f(x) = \ln(x)$ is $f'(x) = \dfrac{1}{x}$

Proof.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\ln(x+h) - \ln(x)}{h} \\
&= \lim_{h\to 0} \frac{\ln \frac{x+h}{x}}{h} \\
&= \lim_{h\to 0} \frac{1}{h} \ln \frac{x+h}{x} \\
&= \lim_{h\to 0} \frac{1}{x} \frac{x}{h} \ln \frac{x+h}{x} \\
&= \lim_{h\to 0} \frac{1}{x} \ln(\frac{x+h}{x})^{\frac{x}{h}} \\
&= \frac{1}{x} \lim_{h\to 0} \ln(1 + \frac{h}{x})^{\frac{x}{h}} \\
&= \frac{1}{x}
\end{aligned}$$

∎

The last part is true because $\lim_{h\to 0} \ln(1 + \dfrac{h}{x})^{\frac{x}{h}} = \ln(e) = 1$.

5 Derivative of multiplication by a constant

Let a be a constant. The derivative of the function $f(x) = au(x)$ is $f'(x) = au'(x)$

Proof.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{au(x+h) - au(x)}{h} \\ &= a \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} \\ &= au'(x) \end{aligned}$$

∎

6 Derivative of the sum of functions

The derivative of the function $f(x) = u(x) + v(x)$ is $f'(x) = u'(x) + v'(x)$

Proof.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{u(x+h) + v(x+h) - u(x) - v(x)}{h} \\ &= \lim_{h\to 0} \frac{u(x+h) - u(x) + v(x+h) - v(x)}{h} \\ &= \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} + \lim_{h\to 0} \frac{v(x+h) - v(x)}{h} \\ &= u'(x) + v'(x) \end{aligned}$$

∎

7 Product Rule

The derivative of the function $f(x) = u(x)v(x)$ is $f'(x) = u(x)v'(x) + u'(x)v(x)$

Proof.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h} \\
&= \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x)v(x) + u(x+h)v(x) - u(x+h)v(x)}{h} \\
&= \lim_{h\to 0} \frac{u(x+h)[v(x+h) - v(x)] - v(x)[u(x+h) - u(x)]}{h} \\
&= u(x) \lim_{h\to 0} \frac{v(x+h) - v(x)}{h} - v(x) \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} \\
&= u(x)v'(x) + u'(x)v(x)
\end{aligned}$$

■

8 Quotient Rule

The derivative of the function $f(x) = \dfrac{u(x)}{v(x)}$ is $f'(x) = \dfrac{u'(x)v(x) - u(x)v'(x)}{v(x)^2}$

Proof.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{u(x+h)}{v(x+h)} - \dfrac{u(x)}{v(x)}}{h} \\
&= \lim_{h\to 0} \frac{v(x)u(x+h) - u(x)v(x+h)}{v(x)v(x+h)h} \\
&= \frac{1}{v(x)} \lim_{h\to 0} \frac{1}{v(x+h)} \lim_{h\to 0} \frac{v(x)u(x+h) - u(x)v(x+h)}{h} \\
&= \frac{1}{v(x)^2} \lim_{h\to 0} \frac{v(x)u(x+h) - u(x)v(x+h)}{h} \\
&= \frac{1}{v(x)^2} \lim_{h\to 0} \frac{v(x)u(x+h) - u(x)v(x+h) + v(x)u(x) - v(x)u(x)}{h} \\
&= \frac{1}{v(x)^2} \lim_{h\to 0} \frac{v(x)[u(x+h) - u(x) - u(x)[v(x+h) - v(x)]}{h} \\
&= \frac{1}{v(x)^2} [v(x) \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} - u(x) \lim_{h\to 0} \frac{v(x+h) - v(x)}{h}] \\
&= \frac{1}{v(x)^2} [u'(x)v(x) - u(x)v'(x)] \\
&= \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2}
\end{aligned}
$$

■

Appendix B Taylor Series

Definition B.1 Taylor Series (Infinite)

The **Taylor Series** of a function $f : \mathbb{R} \to \mathbb{R}$ at a point x^*, assuming $f(x)$ is infinitely differentiable, is defined by:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x^*)}{n!}(x - x^*)^n = f(x^*) + \frac{f'(x^*)}{1!}(x - x^*) + \frac{f''(x^*)}{2!}(x - x^*)^2 + ...$$

Definition B.2 Taylor Series (Infinite, for functions with n variables)

The **Taylor Series** of a function $f : \mathbb{R}^n \to \mathbb{R}$ at a point x^*, assuming $f(x)$ is infinitely differentiable, is defined by:

$$\sum_{i=0}^{\infty} \frac{\nabla^{(i)} f(x^*)}{i!}(x - x^*)^n = f(x^*) + (x - x^*)' \frac{\nabla f(x^*)}{1!} + (x - x^*)' \frac{H(x^*)}{2!}(x - x^*) + ...$$

Definition B.3 nth degree Taylor Polynomial

The **Taylor Polynomial** of a function $f(x)$ at a point x^*, assuming $f(x)$ is n-times differentiable, is defined by:

$$\sum_{i=0}^{n} \frac{f^{(i)}(x^*)}{i!}(x - x^*)^i = f(x^*) + \frac{f'(x^*)}{1!}(x - x^*) + \frac{f''(x^*)}{2!}(x - x^*)^2 + ... + \frac{f^{(n)}(x^*)}{n!}(x - x^*)^n$$

Appendix C Additional Theorems

Theorem C.1 Mean Value Theorem (and Rolle's theorem)

If $f(x)$ is defined in the interval $[a, b]$, then there is a $c \in (a, b)$, such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. The secant line that connects $(a, f(a))$ and $(b, f(b))$ is given by the following equation,

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

Then, we can define the following function

$$g(x) = f(x) - [\frac{f(b) - f(a)}{b - a}(x - a) + f(a)]$$

The last function represents the difference between the secant line for and the original function. Note that $g(a) = g(b) = 0$. Also, there most be a value $c \in (a, b)$, $g'(c) = 0$ (this is known as the Rolle's theorem, you can easily proof this by noting that if $g'(c) \geq 0$ then $g(a) \leq g(b)$ and if $g'(c) \leq 0$ then $g(a) \geq g(b)$; therefore, if $g(a) = g(b)$ it most be that $g'(c) = 0$). Using this result and the last expression,

$$\begin{aligned} g'(c) &= f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \\ f'(c) &= \frac{f(b) - f(a)}{b - a} \end{aligned}$$

■

Theorem C.2 Intermediate Value Theorem

If $f(x)$ is continuous on an intervan $[a, b]$, and $c \in [f(a), f(b)]$, then $\exists x^* \in [a, b]$ such that $f(x^*) = c$.

Proof. tbw. ■

Theorem C.3 Brouwer's Fixed Point Theorem

Suppose that $f : A \to A$ is a continuous function from A to itself. Then f has a **fixed point**. A fixed point is a point x^* such that $f(x^*) = x^*$.

Proof. tbw. See MWG Theorem M.I.1. (page 952). ■

Note, still need to add IFT and envelope theorem.